WITTEN DEFORMATION and WITTEN LAPLACIANS

(Summary)

Calculus on a oriented Riemannian manifold M^n

All spaces $\Omega^q(M)$ are Frechet spaces when equipped with the family of norms $||\cdot||_r$ or with the norms induced from the Sobolev scalar products $(\cdot,)_k$ and the differential operators are continuous w.r. to the Frechet topology.

- $\Omega^r(M)$,. the $\Omega^0(M)-$ module of smooth $r-$differential forms,
- $d_q : \Omega^r(M) \to \Omega^{r+1}(M)$ exterior derivative, order one differential operator ($d(\omega \wedge \omega') = d(\omega) \wedge \omega' + (-1)^{deg\omega} \omega \wedge d\omega'$)
- For X smooth vector field $\iota_X : \Omega^r(M) \to \Omega^{r-1}(M)$ the $X-$contraction. zero order differential operator
- $L_X : \Omega^r(M) \to \Omega^r(M)$ order one differential operator ($\iota_X(\omega\wedge\omega') = \iota_X(\omega)\wedge\omega' + (-1)^{deg\omega}\omega\wedge\iota_X\omega'$)

  $$L_X := d \cdot \iota_X + \iota_X \cdot d$$

  equivalently $L_X(\omega) = d/dt_{t=0}(\varphi_t^*(\omega)), \varphi_t : M \to M$ the flow of X,
- For $\alpha \in \Omega^1,\ \Lambda_\alpha : \Omega^r(M) \to \Omega^{r+1}(M)$. ($\omega \rightsquigarrow \alpha \wedge \omega$) zero order differential operator

Riemannian metric on an closed oriented smooth manifold

given by the collections $*_q : \Omega^q(M) \to \Omega^{n-q}(M)$ zero order differential operators such that

$$*_{n-q} \cdot *_q = (-1)^{q(n-q)} Id$$

and

$$(\omega, \omega') := \int_M \omega \wedge *_q \omega')$$

is a scalar product for any q.

Define

$$d_q^\sharp := (-1)^{n(q+1)+1} *_{n-q} \cdot d_{n-q-1} \cdot *_{q+1} : \Omega^{q+1}(M) \to \Omega^q(M)$$

order one differential operator, the formal adjoint of d_q

$$\Delta_r := d_r^\sharp \cdot d_r + d_{r-1} \cdot d_{r-1}^\sharp = (d + d^\sharp)^2 : \Omega^r(M) \to \Omega^r(M)$$

self-adjoint order 2 (elliptic) differential operator

$$(\omega, \omega')_{2k} := ((I + \Delta)^k \omega, (I + \Delta)^k \omega')$$

a new scalar product for any k. and note that

1. $d \cdot \Delta = \Delta \cdot d$
2. $d^\sharp \cdot \Delta = \Delta \cdot d^\sharp$
3. $* \cdot \Delta = \Delta \cdot *$

Define

1. $\Omega^q(M)_+ := \mathrm{i}mg d_{q-1} \subset \Omega^q(M)$

2. $\Omega^q(M)_- := \mathrm{i}mg d_q^\sharp \subset \Omega^q(M)$

3. $H^q := \ker d_r \cap \ker d_{r-1}^\sharp \subset \Omega^q(M)$

Theorem 0.1 *- All subspaces $\Omega^q(M)_-,\ \Omega^q(M)_+,\ H^q$ are closed subspaces w.r. to the $C^\infty-$topology , and mutually orthogonal w.r. to all scalar products*
and $\Omega^q = \Omega^q(M)_- \oplus H^q \oplus \Omega^q_+$,
- H^q is cannonically isomorphic to de-Rham cohomology,
- The equation $d\omega(t)/dt = -\Delta_q \omega(t)$ in the Frechet space $\Omega^q(M)$ with the initial condition $\omega(0) = \omega \in \Omega^q(M)$ has always unique solution $\omega(t)$ and $Pr(\omega) = \lim_{t\to\infty} \omega(t)$

Any differential operator $A : \Omega^q \to \Omega^{q'}$ has a formal adjoint $A^\sharp : \Omega^{q'}(M) \to \Omega^q(M)$ of the same order defined by the property

$$(A(\omega), \omega') = (\omega, A^\sharp \omega').$$

Additional formal adjoints:

1. $\iota_X^\sharp := (-1)^{\cdots} * \cdot \iota_X \cdot *$
2. $\Lambda\alpha_X^\sharp := (-1)^{\cdots} * \cdot \Lambda_\alpha \cdot *$
3. $L_X^\sharp := (-1)^{\cdots} * \cdot LX \cdot *$
4. $(f\cdot)^\sharp = (f\cdot)$

Proposition 0.2

1. $\Lambda_{df}^\sharp = (-1)^{\cdots} * \Lambda_{df} \cdot * = \iota_{gradf}$

2. $\iota_{gradf}^\sharp = (-1)^{\cdots} * \iota_{gradf} \cdot * = \Lambda_{df}$

3. $\iota_{gradf} \cdot \Lambda_{df} + \Lambda_{df} \cdot \iota_{gradf} = ||gradf||^2 Id = ||df||^2 Id$

4. $L_X^\sharp = (-1)^{\cdots} * \cdot L_X \cdot *$ *and* $L_X + L_X^\sharp$ *is zero order differential operator.*

The sign $(-1)^{\cdots}$ can de derived from "adjunction property" .
Note the zero order differential operators are linear w.r. to the $\Omega^0(M)-$ module structure
Proof: It suffices to check the above statements in coordinates, hence for $M = \mathbb{R}^n$ as indicated below.
For $M = \mathbb{R}^n$ a Riemannian metric is given either by

(1) an $n \times n-$matrix with entries $g_{i,j}$ smooth functions in n-variables, $x_1, x_2, \cdots x_n$ with $g_{i,j} = g_{j,i}$ and $\det g_{i,j} \neq 0$, equivalently the matrix $g^{k,l}$, the inverse of the matrix $g_{r,k}$, in which case for
$\omega = \sum_{1_1, i_2, i_k} \omega_{i_1, i_1, \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k})$ and $\omega' = \sum_{j_1, j_2, \cdots j_k} \omega'_{j_1, j_1, \cdots j_k} dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k})$ one has

$$(\omega, \omega'); = \sum_{I,J} \omega_{i_1, i_2, \cdots i_k} \omega'_{j_1, j_2, \cdots j_k} g^{i_1, j_1} g^{i_2, j_2}, \cdots g^{I_k, j_k}$$

with $I = \{i_1, i_2, \cdots i_k\},\ J = \{j_1, uj_2, \cdots j_k\}$.

(2) the zero order operators $*_q : \Omega^g(\mathbb{R}^n) \to \Omega^{n-q}(\mathbb{R}^n)$ with the property $*_{n-q} \cdot *_q = (-1)^{(n-q)q} Id$. and $(\omega, \omega') := \int \omega \wedge *\omega'$, $\omega, \omega' \in \Omega^q(M)$ is a scalar product for any q.
Indeed, given the operators $*_q' s$ one derives the matriix

$$g^{i,j} := *(dx_i \wedge *dx_j)$$

and given thge matrix $g_{i,j}$, one defines the operators $*_q$ by

$$(*\omega)_{i_1,i_2,\cdots i_{n-q}} = 1/\sqrt{(det(g_{i,j})} \sum_{\sigma\in S_n} \epsilon(\sigma)\omega_{\sigma(1),\sigma(2),\cdots,\sigma(q)} g_{\sigma(q+1),i_1} g_{\sigma(q+2),i_2}, \cdots g_{\sigma(n),i_{n-q}}$$

where S_n is the symmetric group and $\epsilon(\sigma)$ the sign of the permutation σ.

Note also that the formal adjoints $\partial_i^\sharp$, $\Lambda_i^\sharp$ $\iota_i^\sharp$ of the zero order operators ∂_i the component-wise partial derivative w.r. to x_i, $\Lambda_i := \Lambda_{dx_i}$ and $\iota_i := \iota_{\partial/\partial x_i}$ satisfy

1. $$(\partial_j)^\sharp = -(\partial_j) + A$$

   with $A = -1/2\, \partial(\ln|g|)/\partial x_j - \sum_{a,b}(\partial g^{a,b}/\partial x_j)\iota_a^\sharp \cdot \iota_b$ a zero orded differential operator.

2. $$\begin{aligned} \iota_k^\sharp &= \sum_{j=1}^n g_{k,j}\Lambda_j, \;\; \Lambda_j = \sum_{l=1}^n g^{j,l} \;\cdot \iota_l^\sharp \\ \Lambda_{df} &= \sum_{j=1}^n \partial f/\partial x_j \Lambda_j, \;\; \Lambda_{df}^\sharp = \sum_{j=1}^n \partial f/\partial x_j \Lambda_j^\sharp \end{aligned} \tag{1}$$

and

$$d = \sum_{j=1}^n \Lambda_j \cdot \partial_j = \sum_{j=1}^n \partial_j \cdot \Lambda_j$$

$$d^\sharp = \sum_{j=1}^n \partial_j^\sharp \cdot \Lambda_j^\sharp$$

■

Witten deformation

Consider the cochain complex $\Omega^q(M), d_q$ and the multiplication by e^{th}, $e^{tf}; \Omega^q(M) \to \Omega^q(M)$ which induces the isomorphism of cochain complexes

$$e^{tf} : (\Omega^*(M), d_*(t)) \to (\Omega^*(M), d_*)$$

where $d(t) = e^{-tf} d e^{tf} = d + t\Lambda_{df}$ and $d^\sharp(t) := d^\sharp + t\iota_{gradf}$

Witten Laplacians

Define $\Delta_q^f(t) := d^\sharp(t)\cdot d(t) + d(t)\cdot d^\sharp(t)$

Hence by calculations, and in view of Proposition above one has

$$\Delta_q(t) = \Delta_q + t(L_X + L_X^\sharp) + t^2||gradf||^2 Id$$

with $(L_X + L_X^\sharp)$ order zero operator.

Since $\Delta_q(t)$ is a selfadjoint operator of polynomial form with coefficients self-adjoint operators, in view of a Theorem (of Rellich and Kato) one has

Theorem 0.3 *For any q there exists the pairs of analytic real valued resp $q-$ forms valued maps $(\lambda^q_\alpha(t) \in \mathbb{R}, \omega^q_\alpha(t) \in \Omega^q(M))$, with α in a countable collection of indices such that :*

$\Delta_q(t)(\omega_\alpha(t)) = \lambda_\alpha(t)\omega_\alpha(t), ||\omega_\alpha(t)|| = 1$

with $\lambda_\alpha(t)$ exhausting the set of all eigenvalues with multiplicity of $\Delta_q(t)$ and $\omega^q_\alpha(t)$ forming a complete set of orthonormal eigenvectors for the Hilbert space completion of $\Omega^q(M)$. .

Moreover , $\lambda^q_\alpha(t_0) = 0$ for one t_0 implies $\lambda^q_\alpha(t) = 0$ for all t and exactly $dim H^q(M:\mathbb{R})$ eigenbranches $\lambda^q_\alpha(t)$ which are identically zero and all other are strictly positive for any t.

It can be shown (see Haller) that $\lambda^q_\alpha(t) \leq O(t^2)$ and $\lim_{t\to\infty} = \mu \in \mathbb{R}$.

Reference :

1. Perturbation theory for linear operators , T Kato, Springer Verlag edition 1976,

2. Analytic eigenbranches in semiclassical limit, by S. Haller in Complex Anal. and Operator theory Vol 14 , 2020 , 52